Massera's theorem for monotone dynamical systems in three dimensions. [1]

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Abstract. *In this paper we give a variant of the classical second Massera's theorem for three dimensional periodic systems, that satisfy different types of monotonicity assumptions.*

1 Introduction.

Let

$$x' = f(t, x) \tag{1}$$

be a T-periodic system of differential equations. We assume conditions on the function $f : \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}^N$ in order to have existence, uniqueness and continuation of the initial value problem. Our main intention is to ensure when the existence of a solution x bounded in $[0, \infty)$ implies the existence of a T-periodic one.

If $N = 1$, the existence of a solution, bounded in the future implies existence of a T-periodic solution that is the limit of the bounded one. When $N = 2$ then the result is still true but now the periodic solution need not to be the limit of the bounded one. When $N \geq 3$ the result is false unless some more hypotheses are imposed. (See [6] for the case $N = 1$ and $N = 2$ and [8] for an example with $N = 3$.)

In this work we assume some additional hypotheses on equation (1) to have this property for $N = 3$. In concrete we prove that if one solution is bounded to the future and the return map is monotone increasing with respect to an abstract order, then there exists a periodic solution.

This result with the usual order in $\mathbb{R}^3$ allows to work with cooperative and competitive systems coming from the Mathematical Biology. As an application, we prove that existence of a bounded persistent solution implies

[1] AMS subject classification: 37C65, 34C12

[2] Supported by CICYT, PB98-1294, Spain.

the existence of a periodic one. In those situations the equation is not defined in the whole space but in the first octant, however a simple change of variables reduces them to our setting. In this way I can connect my results with those in [9] (see Remark 3.4 for more details).

Since we use of an abstract order in $\mathbb{R}^3$, we can connect these results of Massera type with [12]. In this work, R. A. Smith proves an extension of Massera's theorem by imposing a condition on the derivatives of f, that for $N = 3$ can be interpreted as an abstract order in $\mathbb{R}^3$. This was shown in [10] by Ortega and Sánchez and also that the hypotheses can be relaxed. Therefore this gives a generalization for [12] in three dimensions. Let us observe that the results in [12] hold for any dimension.

To prove these results we do a variant of the Brouwer's lemma for monotone operators. By Brouwer's Lemma we refer to the following result that is a consequence of the translation arcs lemma also due to Brouwer.

Lemma 1.1 *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation preserving homeomorphism without fixed points, then every orbit of*

$$x_{n+1} = f(x_n)$$

tends to infinity.(See [1] for a recent proof)

The idea in this work is to reduce the dimension and to apply the two dimensional case. This idea of reducing the dimension was used several times in the bibliography in order to apply some two dimensional results, see [11], [7] for autonomous cases, [9] and [2] for periodic ones.

The structure of the paper is the following: Section 2 is dedicated to the proof of a variant of the Brouwer's lemma for abstract monotone homeomorphisms in $\mathbb{R}^3$.

Section 3 is dedicated to the concrete application of this Brouwer's lemma, first for cooperative systems, second for competitive ones and thirdly for Smith's systems.

2 A variant of Brouwer's lemma.

Let C be a closed cone in $\mathbb{R}^N$ with nonempty interior, this cone induces an order in $\mathbb{R}^N$ that we will denote by $\leq$. We will say that $x < y$ if $x \leq y$ and $x \neq y$, also $x \ll y$ if $y - x \in \overset{\circ}{C}$.

Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be an homeomorphism, we will say that f is increasing if for every $x \leq y$ then $f(x) \leq f(y)$. This has two consequences:

- For every $x < y$ then $f(x) < f(y)$.
- For every $x \ll y$ then $f(x) \ll f(y)$. (Since f is increasing $f(x + C) \subset f(x) + C$, now since domain invariance $f(x + \overset{\circ}{C})$ is an open set and then

$$f(x + \overset{\circ}{C}) \subset f(x) + \overset{\circ}{C}$$

so the conclusion holds.)

We are going to use the orientation property for homeomorphisms. This can be characterized in terms of topological degree as in chapter IV, proposition 5.12 of [3].

Let us call

$$\mathcal{M}_N = \left\{ f : \mathbb{R}^N \to \mathbb{R}^N \;\Big/\; \begin{array}{c} f \text{ is an increasing orientation} \\ \text{preserving homeomorphism.} \end{array} \right\}$$

Take $f \in \mathcal{M}_N$ and consider the discrete dynamical system

$$x_{n+1} = f(x_n). \tag{2}$$

Theorem 2.1 *Assume $f \in \mathcal{M}_3$, then one of the following properties holds:*

1. *$Fix(f) \neq \emptyset$.*
2. *Every orbit diverges ($\lim_{n\to\infty} | x_n | = +\infty$).*

Before entering into the proof of this theorem we do a simple application of this theorem:

Corollary 2.2 *Assume that $f = g^{-1}$ with $g \in \mathcal{M}_3$, then one of the following properties holds*

1. *$Fix(f) \neq \emptyset$.*
2. *Every semiorbit $\{x_n\}_{n\geq 0}$ is unbounded.*

Proof of the corollary 2.2 from the theorem 2.1. Let $x_o \in \mathbb{R}^3$ and suppose that the semiorbit x_n is bounded, then the omega limit set, L_ω is a nonempty compact invariant set. Hence there exists an orbit that is bounded but now to the past, then last theorem can be applied to $f^{-1} = g$ and f has fixed points. This finishes the proof.

To do the proof of the theorem 2.1, we need some preliminary definitions and results.

Definition 2.3 *Let K be a proper subset of $\mathbb{R}^N$, following the terminology of [13] we will say that K is a C.S.U.C. (closed subset upper closed) if it is closed and satisfies: if $x \in K$ then $y \in K$ for every $y \geq x$.*

The following result gives the structure of this class of sets. For convenience we denote $x = (\tilde{x}, \bar{x}) \in \mathbb{R}^{N-1} \times \mathbb{R}$.

Proposition 2.4 *Let K be a C.S.U.C. then there exists an homeomorphism $\mathcal{F} : \mathbb{R}^N \to \mathbb{R}^N$ such that*

$$\begin{aligned} \mathcal{F}(K) &= \{x \in \mathbb{R}^N / \bar{x} \geq 0\}, \\ \mathcal{F}(\partial K) &= \{x \in \mathbb{R}^N / \bar{x} = 0\}, \\ \mathcal{F}(\overset{\circ}{K}) &= \{x \in \mathbb{R}^N / \bar{x} > 0\}. \end{aligned}$$

Proof. Take $e \in \overset{\circ}{C}$ fixed in this proof, and consider the orthogonal proyection onto $H := e^\perp$, $P : \mathbb{R}^N \to H$. First of all, let us prove that the restriction to the boundary of K is an homeomorphism. It is proved in [4] proposition 2.6 or [13] proposition 1.2 that $P_{|\partial K}$ is a Lipschitz–continuous homeomorphism onto its image. Let us observe that in this case it is onto. To see this, we take some point x_o on the boundary and choose any $v \in H$, then since $e \in \overset{\circ}{C}$, we have $v + \lambda e \gg x_o$ for λ large enough and then $v + \lambda e$ belongs to K. Also for the same reason $v + \lambda e \ll x_o$ for λ small enough ($\lambda \to -\infty$) and then $v + \lambda e$ does not belong to K (otherwise $x_o \in v + \lambda e + \overset{\circ}{C} \subset \overset{\circ}{K}$).

Then for some λ', $v + \lambda' e \in \partial K$. We have seen that for any $v \in H$, we can find a point $v + \lambda' e$ belonging to ∂K such that $P(v + \lambda' e) = v$. So $P_{|\partial K}$ is onto.

Let us go into the construction of $\mathcal{F}$ now. Take $Q = P^{-1}_{|\partial K}$ and take

$$\mathcal{F}(x) = Q(P(x)) + P(x) - x.$$

It is easily checked that $\mathcal{F}$ is an involution such that

$$\mathcal{F}(\partial K) \subset H, \mathcal{F}(H) \subset \partial K.$$

This implies that $\mathcal{F}$ is a homeomorphism from $\mathbb{R}^N$ onto itself that maps homeomorphicaly H onto ∂K. Thus, $\mathbb{R}^N \backslash \partial K$ has two connected components wich are homeomorphic via $\mathcal{F}$ to open halfplanes. If we prove that $\overset{\circ}{K}$ is a connected component of $\mathbb{R}^N \backslash \partial K$, it has to be mapped onto one of the halfplanes of $\mathbb{R}^N \backslash H$. Then, it only rests to compose with an appropiate rotation.

To prove that $\overset{\circ}{K}$ is a connected component, we only have to prove that $\overset{\circ}{K}$ is path connected since every path going out of $\overset{\circ}{K}$ goes through ∂K (a connected component of $\mathbb{R}^N \backslash H \cong \mathbb{R}^N \backslash \partial K$ is path connected). To prove that $\overset{\circ}{K}$ is path connected, we will prove that for every x and y we can find a path connected set containing both points. Take $x,\ y \in \overset{\circ}{K}$ then we can find an $\varepsilon > 0$ such that $x - \varepsilon e, y - \varepsilon e \in \overset{\circ}{K}$ so $x, y \in (x - \varepsilon e + \overset{\circ}{C}) \cup (y - \varepsilon e + \overset{\circ}{C}) \subset \overset{\circ}{K}$ wich is path connected. (This is the union of two convex sets with nonempty intersetion, since λe is in both cones for λ large enough).

Now, we are going to analyze the orientation preserving property. Let K be a C.S.U.C. and let $h : K \to K$ be an homeomorphism (not necessarily monotone) . By the domain invariance theorem, $h(\overset{\circ}{K}) = \overset{\circ}{K}$ and $h(\partial K) = \partial K$. Since $\overset{\circ}{K}$ is homeomorphic to $\mathbb{R}^N$ and ∂K to $\mathbb{R}^{N-1}$ respectively, we have two ways to say when h preserves the orientation.

Proposition 2.5 *The following statement are equivalent:*

1. $h_{|\overset{\circ}{K}} : \overset{\circ}{K} \to \overset{\circ}{K}$ *preserves the orientation.*
2. $h_{|\partial K} : \partial K \to \partial K$ *preserves the orientation.*

Proof. By the last proposition we can assume that

$$K = \{x \in \mathbb{R}^N / \bar{x} \geq 0\},$$

and let $\hat{h}$ be defined by

$$h(\hat{x}, 0) = (\hat{h}(\hat{x}), 0).$$

Now since the orientation property can be defined in terms of topological degree (see [3] IV.5.12), we only have to prove that for any $(\hat{x}, \bar{x})$ with $\bar{x} > 0$,

$$\deg_{(\hat{x},\bar{x})} h_{|\overset{\circ}{K}} = \deg_{\hat{x}} \hat{h}.$$

Let us define $\tilde{h}$ and $\breve{h}$ as

$$\begin{aligned} \tilde{h}(x) &= \begin{cases} h(x) & \bar{x} \geq 0, \\ (\hat{h}(\hat{x}), \bar{x}) & \bar{x} \leq 0, \end{cases} \\ \breve{h}(x) &= (\hat{h}(\hat{x}), \bar{x}), \end{aligned}$$

then

$$\deg_{(\hat{x},\bar{x})} h_{|\overset{\circ}{K}} = \deg_{(\hat{x},\bar{x})} \tilde{h} = \deg_{(\hat{x},-\bar{x})} \tilde{h} = \deg_{(\hat{x},-\bar{x})} \breve{h} = \deg_{\hat{x}} \hat{h}.$$

First and third equalities follow from the local charater of the topological degree. The second equality follows from the independence of the orientation property of the base point (see [3], IV.5.12 again). The last equality follows from the product formula since $\breve{h} = \hat{h} \oplus 1_{\mathbb{R}}$ where $1_{\mathbb{R}}$ is the identity map (see [3], IV.5.13 Exercise 3).

Definition 2.6 *A nonempty subset S of $\mathbb{R}^N$ is called balanced if any two points in S are not strongly comparable. ($x, y \in S \Longrightarrow x \not\ll y$ and $y \not\ll x$)*

Now, we return with the dynamical system (2) with $f \in \mathcal{M}_N$. The following definition was given in [5].

Lemma 2.7 *Let S be an invariant balanced set in $\mathbb{R}^N$ then there exists an invariant C.S.U.C. D such that $S \subset \partial D$.*

Proof. Let us take

$$U = \{x \in \mathbb{R}^N \ / \ \exists (n,s) \in \mathbb{N} \times S \text{ with } f^n(x) \gg s\}$$

and take $D = \bar{U}$. It is not difficult to prove, using the invariance of S, that also U is invariant. Hence D is an invariant C.S.U.C., we only need to show that $S \subset \partial D$.

First let observe that $s \subset d$. In fact $x \in S$, then for $\lambda > 0$ small $x+\lambda e \in U$ so $x \in D$. To show that $S \subset \partial D$ it is sufficient to notice thst $S \cap \overset{\circ}{D}$ then we argue by the way of contradiction, if $x \in \overset{\circ}{D}$ we can take a small positive λ such that $x - \lambda e \in \overset{\circ}{D}$. Now since $D = \bar{U}$ we can take $y_k \to x - \lambda e$, with $y_k \in U$. Then for k large enough $y_k \ll x$, so since f is increasing $x \in U$. This is contradictory with the definition of U , because $f^n(x) \in S$ for all n and S is balanced.

Lemma 2.8 *Let $\{x_n\}_{n\in\mathbb{N}}$ be a nonbalanced orbit. Then one of the following properties holds*

1. $\mid x_n \mid \underset{n\to+\infty}{\to} \infty$
2. $x_n - y_n \underset{n\to+\infty}{\to} 0$ *where $\{y_n\}_{n\in\mathbb{N}}$ is a periodic orbit.*

Proof. Since x_n is not balanced there exist p, q such that $x_p \ll x_q$. Assume that $p < q$, the other case is similar. Then since f is increasing, $\{x_{n(q-p)+p}\}_{n\geq 0}$ is strongly increasing so, if it is bounded it converges to a $q-p$ periodic point ($q - p$ need not to be the minimal period). From here the situation 2 will follow easily. Let us observe that necessarily the sequences

$$\{x_{n(q-p)+p+k}\}_{n\in\mathbb{N}} \tag{3}$$

with $k = 1, 2, ..., q - p - 1$ will also converge. Also, if one of this sequences converges then all converge. The only possibility not to be in the second situation is that all sequences in (3) were unbounded. Since they are all increasing, all converge to infinity, and we are in the first situation.

Proof of theorem 2.1. Let us take finally $f \in \mathcal{M}_3$. Let us assume by the way of contradiction that there is no fixed points and we will show that all orbits are unbounded. Let us take an orbit $\{x_n\}_{n\in\mathbb{Z}}$ and distinguish two cases on whether the orbit is balanced or not.

If $\{x_n\}_{n\in\mathbb{Z}}$ is balanced, using lemma 2.7 we can take D an invariant closed set, upper closed with $x_n \in \partial D$. By proposition 2.4 ∂D is homeomorphic to the plane, and by proposition 2.5, $f_{|\partial D}$ preserves the orientation, so we can apply theorem 0.3 of [9] obtaining that the orbit has no accumulation points on ∂D. By proposition 2.4 the orbit has no accumulation points in $\mathbb{R}^3$, and we finish this case.

If x_n is not balanced, we will apply lemma 2.8 and observe that the situation 2 there is not possible. The argument is that a periodic orbit is always balanced (a nonbalanced invariant set contains always a strict monotone sequence) and we are in the other case.

Remark 2.9 *Theorem 2.1 and corollary 2.2 are also valid if we consider $f : K \to K$, with K be a C.S.U.C. of $\mathbb{R}^3$. However, in the case $K = C$ the results are trivial because the monotonicity implies that the origin is a fixed point.*

3 Applications.

In this section we show how these results can be applied to some specific situations. During this section, let us consider a system (1) where f is continuous and T-periodic in the t variable. We will assume uniquenes and continuation of the initial value problem.

Let us call $f = (f_1, f_2, f_3)$, and $x = (x_1, x_2, x_3)$, we will say that the system (1) is cooperative (resp. competitive) if: for every $i \neq j$, the function f_i is increasing (decreasing) with respect to the x_j variable.

Theorem 3.1 *Assume that $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ and that the system is cooperative. Then one of the following properties holds:*

1. *There exists a T-periodic solution of (1).*

2. *Every solution of (1) tends to infinity*

$$\lim_{t \to +\infty} \mid x(t) \mid = +\infty.$$

Proof. It is shown in [11] that the return map

$$x_o \to x(T, x_o)$$

is in this case an increasing homeomorphism. Also it preserves the orientation since the return map is always isotopic to the identity. Then, we use theorem 2.1 and obtain that if there are no T-periodic solutions,

$$\lim_{n \to \infty} \mid x(nT, x_o) \mid = \infty.$$

Now, the thesis follows from standard arguments.

Corollary 3.2 *Assume that* $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ *and that the system is competitive. Then, one of the following properties holds:*

1. *There exists a* T*-periodic solution of (1).*

2. *Every solution of (1) is unbounded.*

Proof. We are in a similar situation. The inverse of the return map is an increasing homeomorphism, so we have to apply Corollary 2.2 instead of Theorem 2.1.

Now we are going to apply these last results to biological systems. Take $f : \mathbb{R} \times (0, \infty)^3 \to \mathbb{R}^3$, we will say that a solution x is persistent if there exist $\varepsilon > 0$ such that $x_i(t) \geq \varepsilon$ for every $i = 1, 2, 3$ and $t \geq 0$.

Theorem 3.3 *Assume that the system is cooperative or competitive and that there exists a persistent bounded solution. Then there exists a periodic one (this is called a coexistence state).*

Proof. The change of variables $y_i = \ln x_i$, $i = 1, 2, 3$ reduces the problem to a system of one of the last two results.

Remark 3.4 *Theorem 3.3 is connected with the work [9], on the competitive case. Here we do not assume dissipativity, also the monotonicity assumption of* f_i *is only for* $i \neq j$ *and need not to be strict. However there, they do not assume continuation of the solutions and their conclusion is stronger. Assuming dissipativity, it is easy to prove using theorem 3.3 that either we have a periodic solution or for every solution*

$$\lim_{t \to \infty} \inf \min\{x_1(t), x_2(t), x_3(t)\} = 0,$$

instead of the condition obtained in [9]

$$\lim_{t \to \infty} \min\{x_1(t), x_2(t), x_3(t)\} = 0.$$

Following the ideas of [12],[10] we fix a nonsingular symmetric matrix P with two negative eigenvalues and a positive one.

Theorem 3.5 *Assume that there exist a function $\lambda : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ such that*

$$P\frac{\partial f}{\partial x}(t,\xi) + \frac{\partial f}{\partial x}(t,\xi)^* P + \lambda(t,\xi)P \tag{4}$$

is positive semidefinite (resp. negative semidefinite). Then the conclusion of theorem 3.1 (resp. corollary 3.2) holds.

Proof. Following the argument in [10] now the return map (resp. the inverse of the return map) is increasing with respect to an abstract order defined by the cone

$$K = \{\xi \in \mathbb{R}^3 / (P\xi, \xi) \geq 0, (\xi, e_+) \geq 0\}.$$

Here e_+ is an eigenvector corresponding to the positive eigenvalue of P (see [10] for the detail). Now the conclusion goes as in the previous results.

Remark 3.6 *It was shown in [10] that the condition imposed in [12] is more restrictive. In fact, in [12] it is imposed (4) for λ constant and negative. Then these theorems are generalizations of the results there. However, the results in [12] work in any dimension.*

Acknowledgments. I thank E. Norman Dancer, Rafael Ortega and Luis A. Sánchez for many suggestions about proofs and references. I thank also Rafael Ortega for reading a first version of this paper.

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